

Linear Algebra

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9.2 - General Vector Space

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Learning objectives

After this lecture, you should be able to:

1. explain the concept of Euclidean vector space;
2. explain the axioms of vector space;
3. analyze if a given set of vectors is a vector space.

Part 1: General Vector Space

Now we shall extend the concept of a vector by extracting the most important properties of familiar vectors and turning them into axioms.

Thus, when a set of objects satisfies these axioms, they will automatically have the most important properties of familiar vectors, thereby making it reasonable to regard these objects as new kinds of vectors.

What is a vector space?

Let V be an arbitrary nonempty set of objects on which two operations are defined:

- **addition**: a rule associating with each pair of objects in \mathbf{u} and \mathbf{v} and an object in $\mathbf{u} + \mathbf{v}$.
- **scalar multiplication**: a rule associating with each scalar k and each object \mathbf{u} in V , and an object in $k\mathbf{u}$.

V is called a **vector space** if it satisfies the following **six axioms**.

Six axioms of vector space

1. **Closure:** For every $\mathbf{u}, \mathbf{v} \in V$, then:

$$\mathbf{u} + \mathbf{v} \in V \quad \text{and} \quad k\mathbf{u} \in V, \text{ for a scalar } k \in \mathbb{R}$$

2. **Commutativity:** For every $\mathbf{u}, \mathbf{v} \in V$, then: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

3. **Associativity:** For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then:

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

4. **Identity:** For every $\mathbf{u} \in V$, there exist identity $\mathbf{0}$ and scalar 1, such that:

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u} \quad \text{and} \quad 1\mathbf{u} = \mathbf{u}$$

5. **Inverse:** For every $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that:

$$\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u}$$

6. **Distributivity:** For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then:

- $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- $(k + m)\mathbf{w} = k\mathbf{w} + m\mathbf{w}$
- $k(m\mathbf{u}) = (km)\mathbf{u}$

Summarize of axioms

V is a vector space if the followings are satisfied:

For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and a scalars $k, m \in \mathbb{R}$, then:

1. $\mathbf{u} + \mathbf{v} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There exist identity $\mathbf{0}$ and scalar 1, such that:
 $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ and $1\mathbf{u} = \mathbf{u}$
5. There exists $-\mathbf{u} \in V$ such that: $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u}$
6. $k\mathbf{u} \in V$
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{w} = k\mathbf{w} + m\mathbf{w}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

So, how to know that V is a vector space?

Remark. In the definition of vector space, we do not specify the nature of the vectors and the operations.

For instance, **we do not require that:**

- vectors are in \mathbb{R}^n ; or
- the operations of addition and scalar multiplications are not necessarily related to the operations $+$ and \times in \mathbb{R} .

Steps to show that a set V is a vector space:

1. Identify the set V and the objects of V (that will be a vector).
2. Identify the operations “addition” and “scalar multiplication” in V .
3. Check that Axiom 1 is *closed* under addition and scalar multiplication.
4. Check the other five axioms.

Part 2: Example

Example of vector space (1)

\mathbb{R}^n (including \mathbb{R}^2 and \mathbb{R}^3) with the standard operations $+$ and \times , is a vector space.

- $V = \mathbb{R}^n = \{\mathbf{u} = (u_1, u_2, \dots, u_n), \text{ where } u_i \in \mathbb{R}\}$
- *Operations:* the addition and scalar multiplication are defined as:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$k\mathbf{u} = (ku_1, ku_2, \dots, ku_n)$$

- *Closure:* for every $u, v \in \mathbb{R}^n$ and $k \in \mathbb{R}$, then

$$\mathbf{u} + \mathbf{v} \in \mathbb{R}^n \quad \text{and} \quad k\mathbf{u} \in \mathbb{R}^n$$

- It can be verified that Axiom 2- Axiom 6 are satisfied. Check it!

Example of vector space (2)

The space of (2×2) -matrices in \mathbb{R} is a vector space.

- V is the set of (2×2) -matrices with elements in \mathbb{R} .
- The addition and scalar multiplication operations are defined as:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

$$k\mathbf{u} = k \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

- Clearly, $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ are matrices of size 2×2 .
- *Can you verify Axiom 2 - Axiom 6?*

Example of vector space (2) (cont.)

- **Axiom 2 (commutative):**

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$\mathbf{u} + \mathbf{v} \qquad \qquad \qquad = \qquad \qquad \qquad \mathbf{v} + \mathbf{u}$

- **Axiom 3 (identity):** the zero element is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

also:

$$1\mathbf{u} = 1 \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

- **Axiom 4 (inverse):** For $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, then $-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$.
So, $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

- **Axiom 5 (associative)** and **Axiom 6 (distributive)** are also satisfied.

Example of vector space (3)

Vector space of real functions

Let V be a set of functions in \mathbb{R} , namely: $f := \mathbb{R} \rightarrow \mathbb{R}$, with addition and scalar multiplication are defined as follows:

- For $f = f(x)$ and $g = g(x)$ in V and $k \in \mathbb{R}$:

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (kf)(x) = kf(x)$$

Then V is a vector space.

- **Closure:** Note that $(f + g)(x)$ and $(kf)(x)$ are functions in \mathbb{R} . So, $(f + g), (kf) \in V$.
- **Commutative:** $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$
- **Zero identity** is the function: $z(x) = 0$ (all elements are mapped to 0), and it satisfies:

$$f(x) + 0 = 0 + f(x) = f(x)$$

- The **negative** of $f(x)$ is $-f(x)$, meaning that $-f := x \rightarrow -x$, and it satisfies:

$$f(x) + (-f(x)) = 0 = (-f(x)) + f(x)$$

Example of vector space (4)

Vector space of polynomials

Let V be the set of all polynomials of the form:

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

for $x \in \mathbb{R}$, where addition and scalar multiplication are defined as follows:

For $\mathbf{p} = p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $\mathbf{q} = q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$. Then:

$$\mathbf{p} + \mathbf{q} = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n$$

$$k\mathbf{p} = ka_0 + ka_1x + ka_2x^2 + \cdots + ka_nx^n$$

Then, V is a vector space.

Example of vector space (4) (cont.)

- **Closure:** the addition of two polynomials and scalar multiplication of a polynomial result in a polynomial in \mathbb{R} .
- **Commutative:** $p(x) + q(x) = q(x) + p(x)$
- **Identity element:** there is a zero polynomial 0 , s.t. $p(x) + 0 = 0 + p(x) = p(x)$.
- **The negative** of polynomial $p(x)$ is:

$$-p(x) = a_0 - a_1x - a_2x^2 - \dots - a_nx^n$$

so: $p(x) + (-p(x)) = (-p(x)) + p(x) = 0$.

Example of non vector space

Let $V = \mathbb{R}^2$ be the set of objects of form: $\mathbf{u} = (u_1, u_2)$, $u_i \in \mathbb{R}$. Define addition and scalar multiplication as follows:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

$$k\mathbf{u} = (ku_1, k^2 u_2)$$

V is not a vector space.

- Axiom 1 (*closure*) is satisfied, namely:

$$\forall \mathbf{u}, \mathbf{v} \in V, k \in \mathbb{R}, \mathbf{u} + \mathbf{v} \in V \text{ and } k\mathbf{u} \in V$$

- But, Axiom 4 (*identity*) is not satisfied, namely:

$$1\mathbf{u} = 1(ku_1, k^2 u_2) \neq \mathbf{u} \text{ for some } \mathbf{u} \in V$$

For example, take $\mathbf{u} = (1, 2)$. Then: $1\mathbf{u} = (1, 4) \neq \mathbf{u}$.

to be continued...